

A REMARK ON GRADIENT ESTIMATES FOR SPACELIKE MEAN CURVATURE FLOW WITH BOUNDARY CONDITIONS

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ABSTRACT. We prove a gradient estimate for graphical spacelike mean curvature flow with a general Neumann boundary condition in dimension $n = 2$. This then implies that the mean curvature flow exists for all time and converges to a translating solution.

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1. INTRODUCTION

In this paper we obtain gradient estimates for mean curvature flow (MCF) with general Neumann boundary angle conditions in Minkowski space for dimension $n = 2$, leading to existence of the flow for all time and convergence to a translating solution.

In Euclidean space this problem is well studied. In dimension $n = 2$, S. Altschuler and L. Wu [1] demonstrated the Euclidean counterpart of this paper demonstrating that graphical MCF with fixed boundary angles exists for all time and converges to a translating solution. Further gradient estimates were also obtained in higher dimensions by B. Guan [6] demonstrating long time existence, although these depend on the height of the graph and so are not suitable for convergence of the flow, and further estimates for graphs over killing vector fields have been obtained by J. Lira and G. Wanderly [14]. Further results on gradient estimates in Euclidean space have been obtained by G. Huisken [7], A. Stahl [18], V. Wheeler [19][20] and the author [10].

In semi-Riemannian spaces, K. Ecker and G. Huisken [5] demonstrated that MCF (and related flows) may be used to construct prescribed mean curvature hypersurfaces and in higher codimensions G. Li and I. Salavessa [12] showed that MCF may be applied to find when mappings between Riemannian manifolds are topologically trivial (under some curvature conditions). The Dirichlet boundary value problem for such flows in codimension 1 has been studied by K. Ecker [2][3]. The perpendicular Neumann boundary condition was considered by the author in several settings [11][9][8]. A recent article by G. Li, B. Gao and C. Wu [15] dealt exactly with the problem of general graphical angle conditions described below for general dimension n , however the key boundary gradient lemma in this paper is incorrect. Specifically equation (2.9) in that paper appears to come from differentiating the boundary condition in the normal direction into the domain where no such boundary condition holds. Here the author uses methods similar to Altschuler and Wu's [1] to provide an alternative proof for this result in the restricted case of $n = 2$.

Let $\Omega \subset \mathbb{R}^n$ be a compact convex domain with smooth boundary $\partial\Omega$, where we will take $\mathbb{R}^n \subset \mathbb{R}_1^{n+1}$ to be perpendicular to the vector e_{n+1} where $\langle e_{n+1}, e_{n+1} \rangle =$

–1. Let Σ be the cylinder over $\partial\Omega$ in the direction \mathbf{e}_{n+1} . Define γ to be both the outward pointing unit normal to $\partial\Omega$ and μ to be the extension of this to the outward unit normal to Σ .

Let M^n be an n -dimensional disk with boundary ∂M^n and $\mathbf{F} : M^n \times [0, T) \rightarrow \mathbb{R}_1^{n+1}$ be a smooth map such that $\mathbf{F}(\cdot, t)$ is a spacelike embedding of M^n into Minkowski space for all $t \in [0, T)$. Let $\alpha : \Sigma \rightarrow \mathbb{R}$ be a smooth function which will be used to prescribe the boundary angle. Suppose we are given the spacelike smooth initial embedding $\mathbf{F}_0 : M^n \rightarrow \mathbb{R}_1^{n+1}$ then \mathbf{F} moves by MCF with α -Neumann boundary angle condition if

$$(1) \quad \begin{cases} \langle \frac{d\mathbf{F}}{dt}(p, t), \nu(p, t) \rangle = -H(p, t) & \forall (p, t) \in M^n \times [0, T) \\ \mathbf{F}(p, 0) = \mathbf{F}_0(p) & \forall p \in M^n \\ \mathbf{F}(p, t) \subset \Sigma & \forall (p, t) \in \partial M \times [0, T) \\ \langle \nu(p, t), \mu(\mathbf{F}(p, t)) \rangle = \alpha(\mathbf{F}(p, t)) & \forall (p, t) \in \partial M \times [0, T) \end{cases}$$

where ν is a smooth normal to the embedding of $M_t = \mathbf{F}(M^n, t)$. We will assume from now on that \mathbf{F}_0 is smooth and satisfies compatibility conditions, namely that for $p \in \partial M^n$, $\mathbf{F}_0(p) \in \Sigma$ and $\langle \nu(p), \mu(\mathbf{F}_0(p)) \rangle = \alpha$. We remark that the inner product formulation of the first line of (1) is necessary as otherwise (in general) we would require the boundary of M^n to vary with time, as with the usual formulation of MCF, the parametrisation would flow “out of” the interior of Σ .

We will write ∇ for the connection on M_t , ∇^Σ for the connection on Σ , and the ambient connection on \mathbb{R}_1^{n+1} will be denoted $\bar{\nabla}$. The second fundamental form on M and Σ will be written $A(X, Y) =: \langle \bar{\nabla}_X \nu, Y \rangle$ and $A^\Sigma(X, Y) = \langle \bar{\nabla}_X \mu, Y \rangle$ respectively. We observe that when $n = 2$, since Σ is a cylinder, $A^\Sigma(X, Y) = \langle X + \langle X, e_{n+1} \rangle e_{n+1}, Y + \langle Y, e_{n+1} \rangle e_{n+1} \rangle \kappa$ where κ is the curvature of the curve defined by $\partial\Omega \subset \mathbb{R}^2$. We will say that $\partial\Omega \subset \mathbb{R}^2$ is strictly convex if $\kappa > 0$.

We will say α is a *graphical* boundary angle if for all $p \in \Sigma$, $\nabla_{e_{n+1}}^\Sigma \alpha|_p = 0$, that is the boundary angle does not vary in the e_{n+1} direction. If \mathbf{F}_0 is spacelike then we may represent \mathbf{F}_0 as a graph $u_0 : \Omega \rightarrow \mathbb{R}$ initially with the derivative bound $|Du_0| < 1$. If in addition α is graphical, equation (1) is equivalent (by an argument identical to [4, Section 1]) to finding $u : \Omega \times [0, T) \rightarrow \mathbb{R}$ such that

$$(2) \quad \begin{cases} u_t = \sqrt{1 - |Du|^2} D_i \left(\frac{D_i u}{\sqrt{1 - |Du|^2}} \right) & \forall (x, t) \in \Omega \times [0, T) \\ u(x, 0) = u_0(x) & \forall x \in \Omega \\ \gamma^i D_i u(x, t) = \sqrt{1 - |Du|^2} \alpha(x, t) & \forall (x, t) \in \partial\Omega \times [0, T) \end{cases} .$$

We define a translating solution to (1) to be one which stays the same up to reparametrisation and translation depending on time. This may be viewed as a solution of (2) of the form $\tilde{u}(x, t) = \tilde{u}(x, 0) + \lambda t$ for some λ .

A key ingredient to demonstrating uniform parabolicity to equation (2) is finding a gradient estimate, such that there is a constant C depending only on the initial data, α and Ω such that $|Du|(x, t) \leq C < 1$ for all the time the flow exists. Equivalently we require an upper estimate on

$$v := -\langle \nu, e_{n+1} \rangle = \frac{1}{\sqrt{1 - |Du|^2}} .$$

We obtain this estimate in Proposition 6, and as a corollary we obtain the following:

Theorem 1. *Suppose Ω is a smooth strictly convex domain, and the boundary angle prescription function α is graphical. Then any solution to equation (1) starting from smooth spacelike initial data exists for all time and converges uniformly to a translating solution as $t \rightarrow \infty$.*

Proof. The boundary condition in equation (2) is oblique, and as a result of Proposition 6, the flow is uniformly parabolic with a uniform gradient estimate. Therefore, we may apply methods such as in [13, Section VIII.3] to obtain existence for as long as $|u|$ is bounded. We may get an explicit estimate on $|u|$ by observing that the first line in equation (2) gives $u_t = Hv^{-1}$, and so due to Lemma 5 we have $|u_t| < C_H$, immediately implying that at time t

$$(3) \quad |u(x, t)| = C(u_0) + tC_H \quad .$$

We conclude that a solution to (2) exists for all time.

Since we have a gradient estimate that is uniform in time and a height bound of the form (3), both existence of a translating solution \tilde{u} to (2) and the convergence to \tilde{u} may now be seen by following a strong maximum principle argument as in [16, Section 6.2]. Here, we do not rewrite proof, as the arguments in [16] carry over with only trivial modifications. More precisely the only difference is that we obtain the initial linear equation and boundary condition for w on bottom of p340 and top of p341 of [16] from (2), which is quasilinear with a uniformly oblique boundary condition, meaning that an identical equation follows easily by standard methods. Otherwise the proof of existence of a translating solution and convergence to that solution is identical. \square

2. THE BOUNDARY CONDITION

In this section we consider the effect of the condition

$$\langle \nu, \mu \rangle = \alpha$$

where $\alpha \in C^\infty(\Sigma)$ and $|\nabla^\Sigma \alpha| < C_\Sigma$ and Σ is strictly convex.

Lemma 2. *For $p \in \Sigma$ and $W \in T\Sigma \cap TM_t$,*

$$\nabla_W^\Sigma \alpha = A(W, \mu^\top) + A^\Sigma(W, \nu^\Sigma)$$

Proof. We calculate (see also [17, Proposition 2.2][11, Lemma 5.2])

$$\nabla_W^\Sigma \alpha = W(\langle \nu, \mu \rangle) = A(W, \mu^\top) + A^\Sigma(W, \nu^\Sigma)$$

\square

For $p \in \Sigma$ and $X \in T_p \mathbb{R}_1^{n+1}$, we define projections into $T_p M$, $T_p \Sigma$ and $T_p M \cap T_p \Sigma$ by

$$X^\top = X + \langle X, \nu \rangle \nu, \quad X^\Sigma = X - \langle X, \mu \rangle \mu, \quad X^\tau = X - \frac{\langle X, \mu \rangle}{1 + \alpha^2} \mu^\top + \frac{\langle X, \nu \rangle}{1 + \alpha^2} \nu^\Sigma$$

In particular we have

$$(4) \quad e_{n+1} - v\nu = e_{n+1}^\top = e_{n+1}^\tau - \frac{v\alpha}{1 + \alpha^2} \mu^\top$$

and

$$(5) \quad e_{n+1}^\tau = e_{n+1} + v \left(-\nu + \frac{\alpha}{1 + \alpha^2} \mu^\top \right) = e_{n+1} - \frac{v}{1 + \alpha^2} \nu^\Sigma \quad .$$

We recall that $v := -\langle \nu, e_{n+1} \rangle$ where we choose the sign on e_{n+1} so that $v > 0$, and observe the following lemma:

Lemma 3. *At any point in $\Sigma \cap M_t$, we have*

$$\nabla_{\mu^\top} v = \frac{v}{1 + \alpha^2} [\alpha A(\mu^\top, \mu^\top) - A^\Sigma(\nu^\Sigma, \nu^\Sigma)] - \nabla_{e_{n+1}^\tau}^\Sigma \alpha.$$

Proof. We have that at the boundary

$$\nabla_{\mu^\top} v = -A(e_{n+1}^\top, \mu^\top).$$

Using Lemma 2 and equations (4) and (5),

$$\begin{aligned} \nabla_{\mu^\top} v &= A^\Sigma(e_{n+1}^\tau, \nu^\Sigma) - \nabla_{e_{n+1}^\tau}^\Sigma \alpha + \frac{\alpha v}{1 + \alpha^2} A(\mu^\top, \mu^\top) \\ &= -\frac{v}{1 + \alpha^2} A^\Sigma(\nu^\Sigma, \nu^\Sigma) + \frac{\alpha v}{1 + \alpha^2} A(\mu^\top, \mu^\top) - \nabla_{e_{n+1}^\tau}^\Sigma \alpha \end{aligned}$$

where we also used that e_{n+1} is a zero eigenvector of $A^\Sigma(\cdot, \cdot)$. \square

Lemma 4. *At any point in $\Sigma \cap M_t$, we have*

$$\nabla_{\mu^\top} H = \frac{H}{1 + \alpha^2} [\alpha A(\mu^\top, \mu^\top) - A^\Sigma(\nu^\Sigma, \nu^\Sigma) + \nabla_{\nu^\Sigma}^\Sigma \alpha]$$

Proof. We consider $x(t) \in M^n$ such that $F(x(t), t)$ is constrained to lie on the line $p + se_{n+1} \subset \Sigma$ for some $s \in \mathbb{R}$ and $p \in \partial\Omega$. We see that

$$\frac{dF(x(t), t)}{dt} = H\nu + \frac{H}{v} e_{n+1}^\top$$

because $\frac{dF(x(t), t)}{dt} = \lambda e_{n+1}$ where $\lambda \langle e_{n+1}, \nu \rangle = -H$. We may now see

$$\frac{d\nu(x(t), t)}{dt} = \nabla H + \frac{H}{v} \nabla_{e_{n+1}^\top} \nu$$

where we used that under the flow, $\frac{\partial \nu}{\partial t} = \nabla H$ (see [5, Proposition 3.1]).

We now see that since e_{n+1} is a zero eigenvector of $A^\Sigma(\cdot, \cdot)$,

$$\begin{aligned} \frac{d}{dt} \langle \nu(x(t)), \mu(F(x(t), t)) \rangle &= \nabla_{\mu^\top} H + \frac{H}{v} A(e_{n+1}^\top, \mu^\top) \\ &= \nabla_{\mu^\top} H - \frac{H}{v} A^\Sigma(e_{n+1}^\tau, \nu^\Sigma) - \frac{\alpha H}{1 + \alpha^2} A(\mu^\top, \mu^\top) + \frac{H}{v} \nabla_{e_{n+1}^\tau}^\Sigma \alpha \\ &= \nabla_{\mu^\top} H + \frac{H}{1 + \alpha^2} A^\Sigma(\nu^\Sigma, \nu^\Sigma) - \frac{\alpha H}{1 + \alpha^2} A(\mu^\top, \mu^\top) + \frac{H}{v} \nabla_{e_{n+1}^\tau}^\Sigma \alpha \end{aligned}$$

where we used (4) and Lemma 2 to get the second line and (5) to obtain the third.

Since $\frac{d}{dt} \alpha(F(x(t), t)) = \frac{H}{v} \nabla_{e_{n+1}^\tau}^\Sigma \alpha$, the Lemma follows from (5). \square

3. GRADIENT ESTIMATE FOR $n = 2$

We include following for completeness (compare alternative graphical notation proof in [15, Lemma 2.1]):

Lemma 5. *If the boundary angle α is graphical, that is for all $p \in \Sigma$, $\nabla_{e_{n+1}^\tau}^\Sigma \alpha|_p = 0$, then for all time such that the flow exists,*

$$H^2 \leq C_H^2 v^2$$

where $C_H = \sup_{M_0} \frac{|H|}{v}$.

Proof. We have the following well known evolution equations (see e.g. [2, Proposition 2.3, Proposition 2.6][5, Proposition 3.2, Proposition 3.3])

$$(6) \quad \left(\frac{d}{dt} - \Delta \right) H = -H|A|^2, \quad \left(\frac{d}{dt} - \Delta \right) v = -v|A|^2, \quad ,$$

and so on the interior of M_t ,

$$\begin{aligned} \left(\frac{d}{dt} - \Delta \right) \frac{H^2}{v^2} &= -\frac{2|\nabla H|^2}{v^2} - \frac{6H^2|\nabla v|^2}{v^4} + 8\frac{H}{v^3} \langle \nabla v, \nabla H \rangle \\ &= -\left\langle \frac{\nabla H}{H}, \nabla \frac{H^2}{v^2} \right\rangle + 3 \left\langle \frac{\nabla v}{v}, \nabla \frac{H^2}{v^2} \right\rangle \end{aligned}$$

while meanwhile using (5), Lemma 3 and Lemma 4, we see that at the boundary

$$\begin{aligned} \nabla_{\mu^\top} \frac{H^2}{v^2} &= 2\frac{H^2}{v^2} \left[\frac{\nabla_{\nu^\Sigma}^\Sigma \alpha}{1 + \alpha^2} + \frac{\nabla_{e_{n+1}^\tau}^\Sigma \alpha}{v} \right] \\ &= 2\frac{H^2}{v^3} \nabla_{e_{n+1}^\tau}^\Sigma \alpha \\ &= 0 \quad . \end{aligned}$$

Applying the maximum principle gives the result. \square

Similarly to in [1], the restriction to $n = 2$ is now used to estimate the difficult $A(\mu^\top, \mu^\top)$ term in Lemma 3 by H and $\nabla_{e_{n+1}^\tau} v$. This leads to the following gradient estimate:

Proposition 6 (Gradient estimate in dimension 2). *Suppose that $n = 2$, $\partial\Omega$ is strictly convex and the boundary angle is graphical. Then there exists a time independent constant C depending only on M_0 , $\partial\Omega$, α and $\nabla^\Sigma \alpha$ such that for all time that the flow exists,*

$$v \leq C \quad .$$

Proof. We aim to apply the maximum principle to v , and in view of equation (6), all we need to show is that at a large boundary maximum, $\nabla_{\mu^\top} v \leq 0$. We begin by estimating $A(\mu^\top, \mu^\top)$.

Let $p \in M_t \cap \Sigma$ be a boundary maximum of v such that $v(p) \geq C$ where C is to be chosen later and consider e_{n+1}^τ at this point. Since $|e_{n+1}^\tau|^2 = \frac{v^2}{1+\alpha^2} - 1$, we see that by choosing $C > 2\sup_{\partial\Omega} \sqrt{1+\alpha^2}$ we may assume $e_{n+1}^\tau \neq 0$.

We calculate that at p ,

$$\begin{aligned} 0 &= \nabla_{e_{n+1}^\tau} v \\ &= A(e^\top, e_{n+1}^\tau) \\ &= A(e_{n+1}^\tau, e_{n+1}^\tau) - \frac{v\alpha}{1+\alpha^2} A(\mu^\top, e_{n+1}^\tau) \\ &= A(e_{n+1}^\tau, e_{n+1}^\tau) + \frac{v\alpha}{1+\alpha^2} A^\Sigma(\nu^\Sigma, e_{n+1}^\tau) - \frac{v\alpha}{1+\alpha^2} \nabla_{e_{n+1}^\tau}^\Sigma \alpha \\ (7) \quad &= A(e_{n+1}^\tau, e_{n+1}^\tau) - \alpha A^\Sigma(e_{n+1}^\tau, e_{n+1}^\tau) - \frac{v\alpha}{1+\alpha^2} \nabla_{e_{n+1}^\tau}^\Sigma \alpha \end{aligned}$$

where we used equation (4) on the second line, Lemma 2 on the third and (5) on the fourth.

Using (5) and the fact that e_{n+1} is a zero eigenvector of $A^\Sigma(\cdot, \cdot)$, we see that

$$(8) \quad A^\Sigma(\nu^\Sigma, \nu^\Sigma) = A^\Sigma(\nu - \alpha\mu - ve_{n+1}, \nu - \alpha\mu - ve_{n+1}) = (v^2 - 1 - \alpha^2)\kappa$$

and

$$A^\Sigma(e_{n+1}^\tau, e_{n+1}^\tau) = \frac{v^2}{(1 + \alpha^2)^2} A^\Sigma(\nu^\Sigma, \nu^\Sigma) = \frac{v^2(v^2 - 1 - \alpha^2)}{(1 + \alpha^2)^2} \kappa.$$

We define $T = |e_{n+1}^\tau|^{-1} e_{n+1}^\tau = \sqrt{1 + \alpha^2} (v^2 - \alpha^2 - 1)^{-\frac{1}{2}} e_{n+1}^\tau$ and see

$$(9) \quad A^\Sigma(T, T) = \frac{v^2}{1 + \alpha^2} \kappa.$$

Similarly if we define $\sup_{V \in T_p \partial \Omega, |V|=1} \bar{\nabla}_V \alpha = C_\alpha$ then we observe that

$$(10) \quad |\nabla_{e_{n+1}^\tau}^\Sigma \alpha| \leq v \sqrt{\frac{v^2 - 1 - \alpha^2}{1 + \alpha^2}} C_\alpha.$$

Applying (9) and (10) to (7) gives

$$\begin{aligned} A(T, T) &= \frac{\alpha v^2}{1 + \alpha^2} \kappa + \frac{\alpha v}{v^2 - 1 - \alpha^2} \nabla_{e_{n+1}^\tau}^\Sigma \alpha \\ &\geq \frac{\alpha v^2}{1 + \alpha^2} \kappa - \frac{\alpha v^2}{\sqrt{(1 + \alpha^2)(v^2 - 1 - \alpha^2)}} C_\alpha. \end{aligned}$$

We may therefore use Lemma 5 to estimate

$$\begin{aligned} \frac{1}{1 + \alpha^2} A(\mu^\top, \mu^\top) &= H - A(T, T) \\ &\leq C_H v - \kappa \frac{\alpha v^2}{1 + \alpha^2} + \frac{\alpha v^2}{\sqrt{(1 + \alpha^2)(v^2 - 1 - \alpha^2)}} C_\alpha, \end{aligned}$$

which we may now apply along with (8) and (10) to estimate the right hand side of the boundary derivative of v in Lemma 3

$$\begin{aligned} \nabla_{\mu^\top} v &\leq v \left[\alpha C_H v + \frac{\alpha^2 v^2}{\sqrt{(1 + \alpha^2)(v^2 - 1 - \alpha^2)}} C_\alpha - \frac{v^2 - 1}{1 + \alpha^2} \kappa \right] - \nabla_{e_{n+1}^\tau}^\Sigma \alpha \\ &\leq v \left[\alpha C_H v + \frac{\alpha^2 v^2}{\sqrt{(1 + \alpha^2)(v^2 - 1 - \alpha^2)}} C_\alpha + \sqrt{\frac{v^2 - 1 - \alpha^2}{1 + \alpha^2}} C_\alpha - \frac{v^2 - 1}{1 + \alpha^2} \kappa \right] \end{aligned}$$

which is clearly negative for large enough v . In particular, while $v > 2 \sup_{\partial \Omega} \sqrt{1 + \alpha^2}$ we may estimate

$$\nabla_{\mu^\top} v \leq \frac{v^2}{1 + \alpha^2} [(1 + \alpha^2)(\alpha C_H + (2\alpha^2 + 1)C_\alpha + \kappa) - v\kappa]$$

and so for $v \geq \kappa^{-1}(1 + \alpha^2)(\alpha C_H + (2\alpha^2 + 1)C_\alpha + \kappa)$ we see that $\nabla_{\mu^\top} v \leq 0$. The Lemma follows from the maximum principle by choosing

$$C = \max\{2\sqrt{1 + \bar{\alpha}^2}, (1 + \bar{\alpha}^2)(\underline{\kappa}^{-1}\bar{\alpha}C_H + \underline{\kappa}^{-1}(2\bar{\alpha}^2 + 1)C_\alpha + 1), \sup_{M_0} v\},$$

where $\bar{\alpha} = \sup_{x \in \partial \Omega} |\alpha|$ and $\underline{\kappa} = \inf_{x \in \partial \Omega} \kappa > 0$. □

4. REMARKS ON $n \geq 3$

It would be interesting to obtain similar estimates in higher codimension, and we observe that Lemma 5 did not require a dimensional restriction. Clearly the proof in the previous section will no longer hold, and so we must find some other way of estimating $A(\mu^\top, \mu^\top)$ in Lemma 3.

One possible solution observed by B. Guan [6] in the Euclidean case, is to use an extension of the boundary condition itself to obtain a bound. We extend μ smoothly to all of \mathbb{R}_1^{n+1} so that for all $p \in \mathbb{R}_1^{n+1}$, $\bar{\nabla}_{e_{n+1}} \mu|_p = 0$ and for all $q \in \Sigma$, $\bar{\nabla}_\mu \mu|_q = 0$, and define $\tilde{\alpha} = \langle \mu, \nu \rangle$. We may then observe that at the boundary

$$\nabla_{\mu^\top} \tilde{\alpha} = \alpha A^\Sigma(\nu^\Sigma, \nu^\Sigma) + A(\mu^\top, \mu^\top) \quad ,$$

while we also know that at the boundary $\alpha = \tilde{\alpha}$. We may aim to estimate functions such as $v(1 + \tilde{\alpha}^2)^{-\frac{1}{2}}$, which have a negative boundary derivative, as required. However, due to the indefinite metric on the ambient space (as opposed to in definite spaces), the group of isometries fixing a point are noncompact. This implies we must estimate projections with an extra v term, and several such projections appear in the evolution of $\tilde{\alpha}$. The evolution of $\tilde{\alpha}$ reads

$$\left(\frac{d}{dt} - \Delta \right) \tilde{\alpha} = -\tilde{\alpha}|A|^2 - 2h^{ij} \left\langle \bar{\nabla}_{\frac{\partial F}{\partial x^i}} \mu, \frac{\partial F}{\partial x^j} \right\rangle - g^{ij} \left\langle \bar{\nabla}_{\frac{\partial F}{\partial x^i} \frac{\partial F}{\partial x^j}} \mu, \nu \right\rangle$$

where, in general, no signs may be obtained on the last two terms. These must therefore be estimated by $C_1|A|v^2$ and C_2v^3 respectively, and these large powers of v make estimates in general a challenge, and more than can be dealt with purely from the evolution of v .

REFERENCES

- [1] S. J. Altschler and L. F. Wu. Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle. *Calculus of Variations and Partial Differential Equations*, 2:101–111, 1994.
- [2] K. Ecker. Interior estimates and longtime solutions for mean curvature flow of noncompact spacelike hypersurfaces in minkowski space. *Journal of Differential Geometry*, 45:481–498, 1997.
- [3] K. Ecker. Mean curvature flow of spacelike hypersurfaces near null initial data. *Communications in Analysis and Geometry*, 11:181–205, 2003.
- [4] K. Ecker and G. Huisken. Interior estimates for hypersurfaces moving by mean curvature. *Inventiones mathematicae*, 105:547–569, 1991.
- [5] K. Ecker and G. Huisken. Parabolic methods for the construction of spacelike slices of prescribed mean curvature in cosmological spacetimes. *Communications in Mathematical Physics*, 135:595–613, 1991.
- [6] B. Guan. Mean curvature motion of non-parametric hypersurfaces with contact angle condition. In *Elliptic and Parabolic Methods in Geometry*, pages 47–56, Wellesley (MA), 1996. A K Peters.
- [7] Gerhard Huisken. Non-parametric mean curvature evolution with boundary conditions. *Journal of Differential Equations*, 77:369–378, 1989.
- [8] B. Lambert. A note on the oblique derivative problem for graphical mean curvature flow in Minkowski space. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 82(1):115–120, 2012.
- [9] B. Lambert. Construction of maximal hypersurfaces with boundary conditions. ArXiv preprint, submitted to Manuscripta Mathematica, 2014. <http://arxiv.org/abs/1408.5309>.
- [10] B. Lambert. The constant angle problem for mean curvature flow inside rotational tori. *Mathematical Research Letters*, 21(3):537 – 551, 2014.

- [11] B. Lambert. The perpendicular Neumann problem for mean curvature flow with a timelike cone boundary condition. *Transactions of the American Mathematical Society*, 366:3373–3388, 2014.
- [12] G. Li and I.M.C. Salavessa. Mean curvature flow of spacelike graphs. *Mathematische Zeitschrift*, 269(3-4):697–719, 2011.
- [13] G.M. Lieberman. *Second Order Parabolic Differential Equations*. World Scientific Publishing Co. Pte. Ltd., 1996.
- [14] J.H. Lira and G.A. Wampler. Mean curvature flow of Killing graphs. *Transactions of the American Mathematical Society*, 367:4703–4726, 2015.
- [15] G. Li S. Gao and C. Wu. Translating spacelike graphs by mean curvature flow with prescribed angle. *Archive der Mathematik*, 103:499–508, 2014.
- [16] O. Schnürer. Translating solutions to the second boundary value problem for curvature flows. *Manuscripta Mathematica*, 108:319–347, 2002.
- [17] A. Stahl. Convergence of solutions to the mean curvature flow with a Neumann boundary condition. *Calculus of Variations and Partial Differential Equations*, 4:421–441, 1996.
- [18] A. Stahl. Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition. *Calculus of Variations and Partial Differential Equations*, 4:385–407, 1996.
- [19] V.M. Wheeler. Mean curvature flow of entire graphs in a half-space with a free boundary. *Journal für die reine und angewandte Mathematik*, 690:115–131, 2014.
- [20] V.M. Wheeler. Non-parametric radially symmetric mean curvature flow with free boundary. *Mathematische Zeitschrift*, 276(1):281–298, 2014.

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